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TREATMENT OF LAUNDAU-GINZBURG THEORY WITH CONSTRAINTS

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Abstract

Treatment of a singular Lagrangian with constraints using the canonical Hamiltonian approach is studied. We investigate Landau-Ginzburg theory as a constrained system using the Euler-Lagrange equation for the field system and the canonical approach. The equations of motion are obtained as total differential equations in many variables. It is shown that the simultaneous solutions of the Landau-Ginzburg theory with constraints by canonical approach lead to obtaining canonical phase space coordinates and the reduced phase space Hamiltonian without introducing Lagrange multipliers and without any additional gauge fixing condition.

keywords: Lagrangian and Hamiltonian approach, Singular Lagrangian, Landau-Ginzburg theory.

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1 Introduction

Singular Lagrangian systems represent a special case of more general dynamics called constrained systems. A general feature of constrained systems is the existence of constraints in their classical configurations.

The Lagrangian L of any physical system with N degrees of freedom is a function of N generalized coordinates q_i and N generalized velocities $\dot{q}_i$ as well as the time τ,

$$L \equiv (q_i, \dot{q}_i, \tau), \quad i = 1, ..., N.$$

If the velocities can be expressed in terms of the coordinates and the momenta, L is referred to as regular, otherwise, it is singular. Singular Lagrangian systems represent a special case of more general dynamics called constrained systems. A general feature of a constrained system is the existence of its classical configuration.

The basic ideas of the classical treatment and quantization of such systems were initiated and developed by Dirac [1]. He distinguished between two types of constraints; first- and second-classes. In the case of unconstrained systems, the Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics. The first study of Hamilton-Jacobi equations for arbitrary first-order actions was carried out by Santilli [2]. Gitman and Tyutin [3] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. In the recent past, the canonical method based on the Hamilton-Jacobi formulation was developed to investigate singular systems [4, 5, 6, 7, 8]. In this formalism, there is no need to distinguish between first and second constraints as in the Dirac theory [9, 10]. Also, in the canonical method which has been developed by Güler's [11, 12], the equations of motion were written as total differential equations. In Ref. [13], the discrete singular system was treated as a continuous system. Hamiltonian and Lagrangian formulations are used together. The Hamilton-Jacobi formulation of constrained systems has been studied as seen as in Refs. [14, 15, 16]. Moreover, in Refs. [17, 18, 19, ?] Hamilton-Jacobi quantization have been used to obtain the Path integral quantization for several constraint systems. Our aim in this work is to use the Euler-Lagrange equation to treat the system

of a constrained system, the Landau-Ginzburg theory, and to compare the results to those obtained by Hamilton-Jacobi formulation.

The paper is arranged as follows: In section 2, a brief discussion of the canonical Hamiltonian method is given, together with a treatment of a singular system as a continuous system. Next, in section 3, Lanfau Ginzburg theory is treated as a singular constrained field system. Finally, in section 4, several concluding remarks follow.

2 Theoretical framework

In this section, we review the Hamilton-Jacobi formulation of constrained systems [1, 2], which the starting point of this method is to consider the Lagrangian $L \equiv L(q_i, \dot{q}_i, \tau), i = 1, 2, \ldots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \, \partial \dot{q}_j}, \qquad i, j = 1, 2, \ldots, n, \tag{1}$$

of rank $(n-r), r < n$. Then the r momenta are dependent. The generalized momenta P_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-r+1, \ldots, n, \tag{3}$$

The singularity of the system enables us to solve Eq.(2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_a; \tau) \equiv \omega_b. \tag{4}$$

By substituting Eq.(4) into Eq.(3), we obtain the constraints as

$$H'_\mu = p_\mu + H_\mu(\tau, q_i, p_a) = 0, \tag{5}$$

where

$$H_\mu = -\frac{\partial L}{\partial \dot{q}_\mu}\bigg|_{\dot{q}_a \equiv \omega_a}. \tag{6}$$

In this formulation the usual Hamiltonian H_0 is defined as

$$H_0 = -L + p_a \dot{\omega}_a - \dot{q}_\mu H_\mu. \tag{7}$$

Like functions H_μ, the function H_0 is not an explicit function of the velocities $\dot{q}_\nu$. Therefore, the Hamilton-Jacobi function $S(\tau, q_i)$ should satisfy the following set of Hamilton-Jacobi partial differential equations (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha\left(t_\beta,\ q_\alpha,\ P_i = \frac{\partial S}{\partial q_i},\ P_0 = \frac{\partial S}{\partial t_0}\right) = 0, \tag{8}$$

where
$\alpha, \beta = 0, n-r+1, \ldots, n;\qquad a = 1, 2, \ldots, n-r$, and

$$H'_\alpha = p_\alpha + H_\alpha. \tag{9}$$

The canonical equations of motion are given as total differential equations in variables t_β,

$$dq_p = \frac{\partial H'_\alpha}{\partial p_p}\, dt_\alpha, \quad p = 0, 1, \ldots, n; \quad \alpha = 0, n-r+1, \ldots, n, \tag{10}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha, \qquad a = 1, \ldots, n-r, \tag{11}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}\, dt_\alpha, \qquad \alpha = 0, n-r+1, \ldots, n, \tag{12}$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha}\, dt_\alpha\right), \tag{13}$$

where

$$Z \equiv S(t_\alpha, q_a), \tag{14}$$

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10-12) with constraints

$$H'_\alpha(t_\beta,\ q_a,\ P_i) = 0, \qquad \alpha, \beta = 0, n-r+1, \ldots, n. \tag{15}$$

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable if and only if the variations of H'_α vanish identically, that is

$$dH'_\alpha = 0. \tag{16}$$

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

In Ref. [12] the singualr Lagrangian systems are treated as continuous systems. The Euler-Lagrange equation of a singular-Lagrangian system is given as

$$\frac{\partial}{\partial x_\alpha}\left[\frac{\partial L'}{\partial(\partial_\alpha q_\alpha)}\right] - \frac{\partial L'}{\partial q_\alpha} = 0, \qquad \partial_\alpha q_\alpha = \frac{\partial q_\alpha}{\partial x_\alpha}, \tag{17}$$

with constraints

$$dG_\alpha = -\frac{\partial L'}{\partial x_\alpha}\, dt, \tag{18}$$

where L' is the "modified Lagrangian" defined as

$$L'(x_\mu, \partial_\mu q_a, \dot{x}_\mu, q_a) \equiv L(x_\mu, q_a, \dot{q}_a = (\partial_\mu q_a)\dot{x}_\mu); \tag{19}$$

and

$$G_\alpha = H_\alpha\left(x_\mu, q_a, p_a = \frac{\partial L}{\partial q_a}\right). \tag{20}$$

the solution of Eq. (17), together with the constraints equations (18), gives us the solution of the system.

3 The Landau-Ginzburg theory

The Landau-Ginzburg theory gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics, which is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - k\varphi^*\varphi - \frac{1}{4}\lambda(\varphi^*\varphi)^2, \tag{21}$$

where the covariant is given by

$$D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi. \tag{22}$$

and the electromagnetic tensor is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ with the gauge field A^ν. In the landau-Ginzburg theory φ describes the cooper pairs. In usual quantum electrodynamics, we would put $k = m^2$, where m is the effective mass of electron.

3.1 Hamilton-Jacobi formulation of the Landau-Ginzburg theory

The Lagrangian function (21) is singular, since the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \tag{23}$$

is three. The canonical momenta are defined as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \tag{24}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \tag{25}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0\varphi)^* = \dot{\varphi}^* + ieA_0\varphi^*, \tag{26}$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0\varphi) = \dot{\varphi} - i\,e\,A_0\,\varphi, \tag{27}$$

From Eqs. (24), (26) and (27), the velocities $\dot{A}_i$, $\dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i, p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \tag{28}$$

$$\dot{\varphi}^* = p_\varphi - ieA_0\varphi^*, \tag{29}$$

$$\dot{\varphi} = p_{\varphi^*} + ieA_0\varphi. \tag{30}$$

The canonical Hamiltonian H_0 is obtained as

$$H_0 = \frac{1}{4}\, F^{ij} F_{ij} - \frac{1}{2}\, \pi_i \pi^i + \pi^i\, \partial_i A_0 + p_{\varphi^*} p_\varphi + ieA_0 \varphi p_\varphi \\ - ieA_0 \varphi^* p_{\varphi^*} - (D_i\varphi)^*(D^i\varphi) + k\varphi^*\varphi + \frac{1}{4}\lambda(\varphi^*\varphi)^2. \quad (31)$$

Making use of (7) and (8), we find for the set of HJPDE

$$H_0' = \pi_4 + H_0, \quad (32)$$

$$H' = \pi_0 + H = \pi_0 = 0, \quad (33)$$

Therefor, the total differential equations for the characteristic (9-11) obtained as

$$dA^i = \frac{\partial H_0'}{\partial \pi_i}\, dt + \frac{\partial H'}{\partial \pi_i}\, dA^0, \\ = -(\pi^i + \partial_i A_0)\, dt, \quad (34)$$

$$dA^0 = \frac{\partial H_0'}{\partial \pi_0}\, dt + \frac{\partial H'}{\partial \pi_0}\, dA^0 = dA^0, \quad (35)$$

$$d\varphi = \frac{\partial H_0'}{\partial p_\varphi}\, dt + \frac{\partial H'}{\partial p_\varphi}\, dA^0, \\ = (p_{\varphi^*} + ieA_0\varphi)\, dt, \quad (36)$$

$$d\varphi^* = \frac{\partial H_0'}{\partial p_{\varphi^*}}\, dt + \frac{\partial H'}{\partial p_{\varphi^*}}\, dA^0, \\ = (p_\varphi - ieA_0\varphi^*)\, dt, \quad (37)$$

$$d\pi^i = -\frac{\partial H_0'}{\partial A_i}\, dt - \frac{\partial H'}{\partial A_i}\, dA^0, \\ = [\partial_l F^{li} + ie(\varphi^*\partial^i\varphi + \varphi\, \partial_i\varphi^*) + 2e^2 A^i \varphi\varphi^*]\, dt, \quad (38)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H_0'}{\partial A_0}\, dt - \frac{\partial H'}{\partial A_0}\, dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\, p_\varphi]\, dt, \end{aligned} \tag{39}$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H_0'}{\partial \varphi}\, dt - \frac{\partial H'}{\partial \varphi}\, dA^0, \\ &= [(\overrightarrow{D} \cdot \overrightarrow{D}\varphi)^* - k\varphi^* - \frac{1}{2}\lambda \varphi \varphi^{*2} - ieA_0 p_\varphi]\, dt, \end{aligned} \tag{40}$$

and

$$\begin{aligned} dp_{\varphi^*} &= -\frac{\partial H_0'}{\partial \varphi^*}\, dt - \frac{\partial H'}{\partial \varphi^*}\, dA^0, \\ &= [(\overrightarrow{D} \cdot \overrightarrow{D}\varphi) - k\varphi - \frac{1}{2}\lambda \varphi^* \varphi^2 + ieA_0 p_{\varphi^*}]\, dt. \end{aligned} \tag{41}$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \tag{42}$$

which leads to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\, p_\varphi = 0. \tag{43}$$

Taking the total differential of H'', we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi\, dp_\varphi - iep_\varphi\, d\varphi = 0. \tag{44}$$

3.2 Lagrangian formulation of the Landau- Ginzburg theory

Let us write the above Lagrangian in the form

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\left(\frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}\right) \\ &+ (\partial_\mu + ieA_\mu)\varphi^*(\partial^\mu - ieA^\mu)\varphi - k\varphi^*\varphi - \frac{1}{4}\lambda(\varphi^*\varphi)^2, \end{aligned} \tag{45}$$

The canonical momenta are defined as

$$\pi^{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A^{\nu})} = -F^{\mu\nu}, \tag{46}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)} = (D^{\mu}\varphi)^* = \partial^{\mu}\varphi^* + ieA^{\mu}\varphi^* = -H_1, \tag{47}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi^*)} = (D^{\mu}\varphi) = \partial^{\mu}\varphi - ieA^{\mu}\varphi = -H_2, \tag{48}$$

The singular Lagrangian in Eq.(18) can be treated as a continuous system by introducing

$$A_{\nu} = A_{\nu}(x_{\mu}, \varphi, \varphi^*),\ \varphi = \varphi(x_{\mu}),\ \varphi^* = \varphi^*(x_{\mu}). \tag{49}$$

Let us define the four-dimensional derivative of A_{ν} as

$$\frac{\partial A_{\nu}}{\partial x_{\mu}} \equiv \frac{dA_{\nu}}{dx_{\mu}} = \partial_{\mu} A_{\nu} + \frac{\partial A_{\nu}}{\partial \varphi}\frac{\partial \varphi}{\partial x_{\mu}} + \frac{\partial A_{\nu}}{\partial \varphi^*}\frac{\partial \varphi^*}{\partial x_{\mu}} \tag{50}$$

The modified Lagrangian $\mathcal{L}'$ becomes

$$\begin{aligned} \mathcal{L}' = \frac{1}{4}\Big[&\partial_{\mu}A_{\nu} + \frac{\partial A_{\nu}}{\partial\varphi}\,\partial_{\mu}\varphi + \frac{\partial A_{\nu}}{\partial\varphi^*}\,\partial_{\mu}\varphi^* - \partial_{\nu}A_{\mu} - \frac{\partial A_{\mu}}{\partial\varphi}\,\partial_{\nu}\varphi - \frac{\partial A_{\mu}}{\partial\varphi^*}\,\partial_{\nu}\varphi^*\Big] \\ \times \Big[&\partial^{\mu}A^{\nu} + \frac{\partial A^{\nu}}{\partial\varphi}\,\partial^{\mu}\varphi + \frac{\partial A^{\nu}}{\partial\varphi^*}\,\partial^{\mu}\varphi^* - \partial^{\nu}A^{\mu} - \frac{\partial A^{\mu}}{\partial\varphi}\,\partial^{\nu}\varphi - \frac{\partial A^{\mu}}{\partial\varphi^*}\,\partial^{\nu}\varphi^*\Big] \\ &+ (\partial_{\mu} + ieA_{\mu})\varphi^*(\partial^{\mu} - ieA^{\mu})\varphi - k\varphi^*\varphi - \frac{1}{4}\lambda(\varphi^*\varphi)^2 \end{aligned} \tag{51}$$

The Euler-Lagrangian equation for the continuous system (13), for $q_{\alpha} \equiv x_{\mu}, \varphi, \varphi^*$ and $q_a \equiv A_{\nu}$, becomes

$$\frac{\partial}{\partial x_{\mu}}\left[\frac{\partial \mathcal{L}'}{\partial(\partial_{\mu}A_{\nu})}\right] + \frac{\partial}{\partial\varphi}\left[\frac{\partial \mathcal{L}'}{\partial(\frac{\partial A_{\nu}}{\partial\varphi})}\right] + \frac{\partial}{\partial\varphi^*}\left[\frac{\partial \mathcal{L}'}{\partial(\frac{\partial A_{\nu}}{\partial\varphi^*})}\right] - \frac{\partial \mathcal{L}'}{\partial A_{\nu}} = 0, \tag{52}$$

With the modified Lagrangian $\mathcal{L}'$, Eq.(26) takes the form

$$\partial_{\mu}F^{\mu\nu} + ie(\varphi^*\partial^{\nu}\varphi - \varphi\,\partial^{\nu}\varphi) + 2e^2A^{\nu}\varphi^*\varphi = 0. \tag{53}$$

Equation (27) is the first set of Euler-Lagrange equations obtained from the standard Lagrangian formulation; it is equivalent to the equations of motion obtained from the canonical method[9]. The second set of Euler-Lagrange equations of the standard Lagrangian formulation is obtained by using the constraint equations (14), that is,

$$dG_1 = -\frac{\partial \mathcal{L}'}{\partial \varphi} dx_\mu. \tag{54}$$

G_1 is obtained from the Hamiltonian formulation, Eq.(5):

$$G_1 \equiv H_1 = -(\partial^\mu \varphi^* + ieA^\mu \varphi^*); \qquad and \qquad d\varphi^* = \frac{\partial \varphi^*}{\partial x_\mu} dx_\mu$$

Thus, Eq. (28) becomes

$$(\overrightarrow{D} \cdot \overrightarrow{D}\varphi)^* - ie(2A^\mu \partial_\mu \varphi^* + \varphi^* \partial_\mu A^\mu) - k\varphi^* - \frac{1}{2}\lambda\varphi\varphi^{*2} = 0. \tag{55}$$

Similarly, from Eq.(5), we have

$$G_2 \equiv H_2 = -(\partial^\mu \varphi - ieA^\mu \varphi) \qquad and \qquad d\varphi = \frac{\partial \varphi}{\partial x_\mu} dx_\mu$$

. Then, by using Eq.(14), we get

$$dG_2 = -\frac{\partial \mathcal{L}'}{\partial \varphi^*} dx_\mu, \tag{56}$$

above equation becomes

$$(\overrightarrow{D} \cdot \overrightarrow{D}\varphi) + ie(2A^\mu \partial_\mu \varphi + \varphi\, \partial_\mu A^\mu) - k\varphi - \frac{1}{2}\lambda\varphi^* \varphi^2 = 0. \tag{57}$$

Equations (29) and (31) are the second set of Euler-Lagrange equations of the standard formulation.

4 Conclusion

The Lagrangian of Landau-Ginzburg theory gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics.

This system is studied as a singular Lagrangian using the Euler-Lagrange equation and the canonical Hamiltonian approach (Hamilton-Jacobi approach). The system is treated as a continuous field system with constraints. It is shown that this treatment is in exact agreement with the general approach. Our formalism is a mixture of the Hamiltonian and Lagrangian formulations. In the general approach, the constraint equations can be obtained from the Euler-Lagrange equations; whereas in the treatment of singular Lagrangian as fields, the constraints can be determined from Eq. (14), which is obtained with the help of the canonical Hamiltonian formalism. The equations of motion are obtained as partial differential equations, which are equivalent to those equations obtained from the canonical Hamiltonian approach.

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